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Relativistic Green functions in a plane-wave gravitational background

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Abstract

We consider a massive relativistic particle in the background of a gravitational plane wave. The corresponding Green functions for both spinless and spin-$\frac{1}{2}$ cases, previously computed by Barducci and Giachetti (2005 *J. Phys. A: Math. Gen.* **38** 1615), are reobtained here by alternative methods, as for example, the Fock–Schwinger proper-time method and the algebraic method. In analogy with the electromagnetic case, we show that for a gravitational plane-wave background a semiclassical approach is also sufficient to provide the exact result, though the Lagrangian involved is far from being a quadratic one.

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1. Introduction

Green functions are basic ingredients in quantum theories. Particularly, they are of great importance in the computation of scattering amplitudes, as well as atomic energy levels. However, only a few exact results for Green functions of relativistic particles or even non-relativistic ones under the influence of external fields are available in the literature. Of particular importance, among the methods of obtaining Green functions is the so-called Fock–Schwinger method. It was introduced in the context of relativistic quantum field in 1951 by Schwinger [2]. It has since been employed mainly in relativistic problems such as the calculation of bosonic [3] and fermionic [4–7] Green functions in external fields. The most common exact solutions found in the literature are those involving the non-relativistic Coulomb potential [9], the relativistic Dirac–Coulomb potential [10], a constant and uniform electromagnetic field as well as the electromagnetic field of a linearly polarized plane wave [2, 11] or even particular combinations of constant fields and plane-wave fields [3, 12].

In the presence of an external gravitational background the problem becomes extremely complicated and only a few solutions using algebraic techniques or mode summation method are known. In this context, a simplifying assumption is to consider a weak gravitational

[1] In memoriam.

field. Using the path integral and external source methods it is possible to calculate the Feynman propagator in the case when the gravitational field is that of a plane wave [13, 14]. Recently, Barducci and Giachetti [1] have considered the wave equations for spin-0 and spin-$\frac{1}{2}$ particles in a weak external plane-wave gravitational field. They obtained for the wavefunctions Volkov-type solutions [16]. A similar ansatz for the Green functions is verified to be correct. These results have a resemblance to those for a charged particle in a plane-wave external electromagnetic field. Since the latter case has been treated in a deductive manner by the proper-time method [2], it is interesting to apply this technique for the case of a weak background of a gravitational plane wave.

The purpose of this paper is to provide many alternative methods for computing the above-mentioned Green functions. This work is organized as follows: in section 2 we obtain the Green function for a spin-0 massive particle in the presence of a weak background of a gravitational plane wave by using the Fock–Schwinger method. In section 3, we apply this same method and obtain the corresponding Green function for a spin-$\frac{1}{2}$ particle. Then, in section 4, we show that these solutions may be constructed in a much simpler manner by an operator technique. In section 5, we reobtain the bosonic Green function by trying a convenient ansatz. In section 6, we show that a path integral semiclassical approach is sufficient to yield the exact results, even though the Lagrangian involved is far from being a quadratic one. Finally, section 7 is left for conclusions and final remarks.

2. Proper-time method for a scalar particle

In the linear approximation the gravitational field is described by a small perturbation to the flat metric, namely

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x), \tag{1}$$

where $\eta_{\mu\nu}$ is the flat space metric and $h_{\mu\nu}$ is a small perturbation. The imposition of harmonic condition in the linear approximation gives

$$\partial_\mu h^\mu{}_\nu(x) = \tfrac{1}{2}\partial_\nu h(x). \tag{2}$$

It is convenient to choose a restriction of the harmonic gauge and impose the conditions

$$\partial_\mu h^\mu{}_\nu = 0, \qquad h(x) = 0. \tag{3}$$

Finally, the linearized gravitational field is taken to be the one produced by a wave of arbitrary spectral composition and polarization properties, but propagating in a fixed direction so that

$$h_{\mu\nu}(x) = a_{\mu\nu} F(\xi), \tag{4}$$

where $\xi = n \cdot x$ and the propagation vector is a light-like one, which satisfies the condition $n^2 = 0$, and F is an arbitrary function. The conditions imposed on $h_{\mu\nu}$ give

$$n_\mu a^{\mu\nu} = 0, \qquad \mathrm{Tr}\, a = 0. \tag{5}$$

We note that linearization implies that terms involving second and higher powers of a are to be dropped. In the following calculations it may be convenient sometimes not to do this in the intermediate stages. With these assumptions, the Green function for a scalar particle in a weak external linearized gravitational field satisfies the equation [13]

$$[\partial^\mu\partial_\mu - h^{\mu\nu}\partial_\mu\partial_\nu + m^2]G(x-y) = -\delta(x-y), \tag{6}$$

where $h_{\mu\nu}(x)$ is of the form described previously. In order to apply the proper-time method we write the Green function in the form

$$G(x', x'') = -\mathrm{i}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}m^2 s}\langle x'|\,\mathrm{e}^{-\mathrm{i}sH}|x''\rangle, \tag{7}$$

where the proper-time Hamiltonian is given by

$$H = -p^2 + p^\mu a_{\mu\nu} p^\nu F(\xi), \tag{8}$$

with $p^\mu = \mathrm{i}\partial^\mu$ and

$$[x^\mu, p^\nu] = -\mathrm{i}\eta^{\mu\nu}. \tag{9}$$

We adopt Schwinger's notation in which unprimed quantities are used for operators while primed quantities are used for the corresponding eigenvalues. In this sense, the eigenvalue equation for the operator x^μ is written as

$$x^\mu|x'\rangle = x'^\mu|x'\rangle, \tag{10}$$

where, for simplicity, we omitted indices in the eigenvector $|x'\rangle$. Next, given an operator O, we introduce the operator $O(s)$ defined as

$$O(s) = \mathrm{e}^{\mathrm{i}sH} O\, \mathrm{e}^{-\mathrm{i}sH}, \tag{11}$$

which satisfies the Heisenberg-like equation of motion

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s} O(s) = [O(s), H]. \tag{12}$$

Of course, $O(0) = O$, so that p^μ, x^μ, etc, mean the same as $p^\mu(0)$, $x^\mu(0)$ and so on. Hence, we have

$$x^\mu(s)(\mathrm{e}^{\mathrm{i}sH}|x'\rangle) = x'(\mathrm{e}^{\mathrm{i}sH}|x'\rangle), \tag{13}$$

so that we conveniently define

$$|x's\rangle = \mathrm{e}^{\mathrm{i}sH}|x'\rangle. \tag{14}$$

Note that from (9) and (11) we also have

$$[x^\mu(s), p^\nu(s)] = -\mathrm{i}\eta^{\mu\nu}. \tag{15}$$

Using this 'Heisenberg picture', the Green function can be cast into the form

$$G(x', x'') = -\mathrm{i}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}m^2 s}\langle x's|x''0\rangle, \tag{16}$$

where the 'Schrödinger-like' propagator $\langle x's|x''0\rangle$ satisfies the differential equation

$$\mathrm{i}\partial_s\langle x's|x''0\rangle = \langle x's|H|x''0\rangle \tag{17}$$

submitted to the initial condition

$$\lim_{s\to 0}\langle x's|x''0\rangle = \delta(x' - x''). \tag{18}$$

Schwinger's method for computing $\langle x's|x''0\rangle$ basically consists in the following steps: (i) we first solve the Heisenberg equations for operators $p^\mu(s)$ and $x^\mu(s)$ and write the proper-time Hamiltonian in terms of $x^\mu(s)$ and $x^\mu(0)$, instead of $x^\mu(0)p^\mu(0)$; (ii) then, using the commutator between $x^\mu(s)$ and $x^\mu(0)$ we write conveniently this Hamiltonian in a ordered form in time s, namely with operators $x^\mu(s)$ in all terms put on the left-hand side, so that equation (17) can be immediately integrated in s to yield $\langle x's|x''0\rangle = C(x', x'') \exp\left\{-\mathrm{i}\int^s F(x', x''; s')\mathrm{d}s'\right\}$, where $F(x', x''; s') := \langle x's|H|x''0\rangle/\langle x's|x''0\rangle$; (iii) finally, the integration constant $C(x', x'')$ is obtained by imposing the constraints

$$\begin{aligned} \langle x's|p^\mu(s)|x''0\rangle &= \mathrm{i}\frac{\partial}{\partial x'_\mu}\langle x's|x''0\rangle \\ \langle x's|p^\mu(0)|x''0\rangle &= -\mathrm{i}\frac{\partial}{\partial x''_\mu}\langle x's|x''0\rangle \end{aligned} \tag{19}$$

as well as the initial condition (18). A pedagogical introduction of Schwinger's method can be found in [17–19] (for other applications to non-relativistic problems, see [20–24]).[1] The Heisenberg equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}s}x^\mu(s) = 2\left[\eta^\mu_\nu - a^\mu_\nu F(\xi(s))\right]p^\nu(s) \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}p^\mu(s) = n^\mu F'(\xi(s))p^\alpha(s)a_{\alpha\beta}p^\beta(s), \tag{21}$$

where in the last equation F' means $\mathrm{d}F/\mathrm{d}\xi$. In order to solve previous equations for $x^\mu(s)$ and $p^\mu(s)$, observe initially that they imply the following ones:

$$\frac{\mathrm{d}}{\mathrm{d}s}\xi(s) = 2n_\mu p^\mu(s) =: 2n\cdot p(s), \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(n\cdot p(s)) = 0, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(p_\mu(s)a^{\mu\nu}p_\nu(s)) = 0. \tag{24}$$

Equations (22) and (23) lead to

$$n\cdot p(s) = \frac{\xi(s)-\xi(0)}{2s} \tag{25}$$

and equation (24) allows us to write

$$p^\mu(s)a_{\mu\nu}p^\nu(s) = C, \tag{26}$$

where C is an operator that does not depend on s. In order to integrate equation (21), we multiply and divide its rhs by $\mathrm{d}\xi/\mathrm{d}s = 2n\cdot p$ to get

$$\frac{\mathrm{d}}{\mathrm{d}s}p^\mu(s) = \frac{n^\mu \frac{\mathrm{d}F}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}s}C}{2n\cdot p} = \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{n^\mu F(\xi(s))C}{2n\cdot p}\right\}, \tag{27}$$

where we omitted the argument s in $2n\cdot p$ or $\mathrm{d}\xi/\mathrm{d}s$ since these quantities are constants of motion and used the fact that $[n\cdot p,\xi] = \mathrm{i}n^2 = 0$. We also omitted the argument of $\mathrm{d}F/\mathrm{d}\xi$ and used that C is an s-independent operator. A direct integration of the above equation leads to

$$p^\mu(s) = \frac{n^\mu F(\xi(s))C}{2n\cdot p} + D^\mu, \tag{28}$$

where D^μ is a constant operator that satisfies

$$n_\mu D^\mu = n_\mu p^\mu = \frac{\xi(s)-\xi(0)}{2s}. \tag{29}$$

For simplicity, from now on, we shall omit indices and a matrix notation will be assumed, so that the previous equation is written simply as

$$p(s) = \frac{nF}{2n\cdot p}C + D. \tag{30}$$

Two relations involving the constant operators C and D can be written from our previous results. Inserting (28) into (26) and computing $p^\mu p_\mu$ from the last equation, we obtain,

[1] Schwinger's method for the non-relativistic oscillator was developed independently by M Goldberger and M Gellmann in 1951 in the context of statistical mechanics [25].

respectively,

$$C = DaD \tag{31}$$

$$p^2 = D^2 + CF. \tag{32}$$

Hence, from equations (26) and (32), the proper-time Hamiltonian (8) can be written as

$$H = -D^2. \tag{33}$$

Then, inserting (30) into (20) and integrating in s following a procedure analogous to that used to obtain $p(s)$, we get

$$x(s) - x(0) = \frac{A(\xi(s)) - A(\xi(0))}{2n \cdot p}\left(\frac{nC}{n \cdot p} - 2aD\right) + 2Ds, \tag{34}$$

where we defined A by

$$F(\xi) = \frac{\mathrm{d}A(\xi)}{\mathrm{d}\xi}. \tag{35}$$

Solving the previous equation for D, we obtain

$$D = M\frac{x(s) - x(0)}{2s} - \frac{nC}{2n \cdot p}\frac{A(\xi(s)) - A(\xi(0))}{\xi(s) - \xi(0)} + \mathcal{O}(a^2), \tag{36}$$

where

$$M := \left[I - a\frac{A(\xi(s)) - A(\xi(0))}{\xi(s) - \xi(0)}\right]^{-1}. \tag{37}$$

Equations (31) and (36) allow us to write constant C in the form

$$C = \frac{1}{4s^2}(x(s) - x(0))aM^2(x(s) - x(0)) + \mathcal{O}(a^2). \tag{38}$$

Using equations (33) and (36), and keeping terms only up to order a, the proper-time Hamiltonian in the weak field approximation takes the form

$$H = -\frac{1}{4s^2}(x(s) - x(0))M(x(s) - x(0)). \tag{39}$$

To write the above proper-time Hamiltonian in the appropriate s-ordered form we note the following commutation relations:

$$[\xi(0), x^\mu(s)] = [\xi(s) - 2n \cdot p(s), x^\mu(s)] = -2\mathrm{i}n^\mu s, \tag{40}$$

$$[M^{\mu\lambda}x_\lambda(s), x_\mu(0)] = 2s\left[D^\mu + \frac{Cn^\mu}{2n \cdot p}\frac{A(\xi(s)) - A(\xi(0))}{\xi(s) - \xi(0)}, x_\mu(0)\right] = 8\mathrm{i}s. \tag{41}$$

Integrating equation $\mathrm{i}\partial_s\langle x's|x''0\rangle = \langle x's|H|x''0\rangle$, we get

$$\langle x's|x''0\rangle = \frac{\Phi(x', x'')}{s^2}\exp\left\{-\frac{\mathrm{i}}{4s}(x' - x'')M(\xi', \xi'')(x' - x'')\right\}, \tag{42}$$

where

$$M(\xi', \xi'') = \left[I - a\frac{A(\xi') - A(\xi'')}{\xi' - \xi''}\right]^{-1} \tag{43}$$

and $\Phi(x', x'')$ is an s-independent quantity to be determined by imposing constraints (19). Let us impose the first constraint written in (19). To evaluate its rhs we need to compute

$\partial'\langle x's|x''0\rangle$, while to evaluate its lhs we need the s-ordered matrix element of operator $p(s)$. Differentiating equation (42), we obtain

$$\mathrm{i}\frac{\partial'\langle x's|x''0\rangle}{\langle x's|x''0\rangle} = \mathrm{i}\frac{\partial'\Phi}{\Phi} + M\frac{(x'-x'')}{2s} + \frac{1}{4s}(x'-x'')\partial' M(\xi',\xi'')(x'-x''), \tag{44}$$

where

$$\partial' M(\xi',\xi'') = \frac{n}{\xi'-\xi''}\left[F(\xi') - \frac{A(\xi')-A(\xi'')}{\xi'-\xi''}\right] a M^2. \tag{45}$$

Using the solution for $p(s)$, obtained from equations (30) and (31), a straightforward calculation yields

$$\partial'\Phi(x',x'') = 0. \tag{46}$$

A similar calculation gives

$$\partial''\Phi(x',x'') = 0. \tag{47}$$

Hence $\Phi(x',x'')$ is a constant denoted simply by Φ. This constant may be evaluated by imposing the initial condition (18). However, the explicit calculation is very similar to that verifying the consistency condition in section 5, so that we avoid it here. The result is given by

$$\Phi = \frac{\mathrm{i}}{(4\pi)^2}. \tag{48}$$

Collecting all previous results, we finally obtain

$$G(x',x'') = \frac{1}{16\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}m^2 s}\exp\left\{-\frac{\mathrm{i}}{4s}(x'-x'')\left[I - a\frac{A(\xi')-A(\xi'')}{\xi'-\xi''}\right]^{-1}(x'-x'')\right\}. \tag{49}$$

As expected, if we take in the above expression the limit $a \to 0$, we recover the well-known free-particle propagator in the Minkowski space.

3. Proper-time method for a Dirac particle

For a spin-$\frac{1}{2}$ particle the Green's function satisfies the equation

$$\left(\mathrm{i}\gamma^\mu\partial'_\mu - \frac{\mathrm{i}}{2}F(\xi')a^{\mu\nu}\gamma_\mu\partial'_\nu - m\right) S_F(x',x'') = \delta(x'-x''), \tag{50}$$

where γ^μ are the usual Dirac matrices. As it is common in problems involving fermion Green functions in external fields, we conveniently define $\Delta_F(x',x'')$ as follows:

$$S_F(x',x'') := \left(\mathrm{i}\gamma^\mu\partial'_\mu - \frac{\mathrm{i}}{2}F(\xi')a^{\mu\nu}\gamma_\mu\partial'_\nu + m\right)\Delta_F(x',x''). \tag{51}$$

As a consequence, $\Delta_F(x,y)$ satisfies a boson-like second-order differential equation, namely

$$\left(\partial'^\mu\partial'_\mu - F(\xi')a^{\mu\nu}\partial'_\mu\partial'_\nu + \frac{\mathrm{i}}{2}\frac{\mathrm{d}F}{\mathrm{d}\xi'}\sigma_{\mu\nu}n^\mu a^{\nu\rho}\partial'_\rho + m^2\right)\Delta_F(x',x'') = -\delta(x'-x''), \tag{52}$$

where $\sigma_{\mu\nu} = (\mathrm{i}/2)[\gamma_\mu,\gamma_\nu]$ and we neglected terms quadratic in a. Using the notation introduced previously, $\Delta_F(x',x'')$ can be written as

$$\Delta_F(x',x'') = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}m^2 s}\langle x'|\,\mathrm{e}^{-\mathrm{i}Hs}|x''\rangle = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}m^2 s}\langle x's|x''0\rangle, \tag{53}$$

where now the corresponding proper-time Hamiltonian can be written in the form

$$H = -p^2 + papF(\xi) + \frac{1}{2}pa\sigma n\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}, \tag{54}$$

where we are following the same notation as before, namely primed quantities are eigenvalues while unprimed ones are operators. In writing the last equation we also used the fact that $[pap, F(\xi)] = 0$ and $[pa\sigma n, \mathrm{d}F(\xi)/\mathrm{d}\xi] = 0$.

From now on, whenever it does not cause any confusion, we shall omit the argument s from the operators involved as well as the argument of F and its derivatives $\mathrm{d}F/\mathrm{d}\xi$ and $\mathrm{d}^2F/\mathrm{d}\xi^2$. With this in mind, the Heisenberg equations of motion for the operators x and p are given, respectively, by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 2(I - aF)p - \frac{1}{2}a\sigma n\frac{\mathrm{d}F}{\mathrm{d}\xi}, \tag{55}$$

$$\frac{\mathrm{d}p}{\mathrm{d}s} = n\left(pap\frac{\mathrm{d}F}{\mathrm{d}\xi} + \frac{1}{2}pa\sigma n\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}\right). \tag{56}$$

In this case, the constants of motion are

$$n\cdot p = \frac{\xi(s) - \xi(0)}{2s}, \tag{57}$$

$$C_1 = pap, \tag{58}$$

$$C_2 = pa\sigma n. \tag{59}$$

Integrating the equation of motion for p we have

$$p(s) = \frac{n}{2n\cdot p}\left(C_1F + \frac{1}{2}C_2\frac{\mathrm{d}F}{\mathrm{d}\xi}\right) + D_f, \tag{60}$$

where D_f is a constant. For future convenience, we use the previous equation to write p^2 in the form

$$p^2 = D_f^2 + C_1F + \frac{1}{2}C_2\frac{\mathrm{d}F}{\mathrm{d}\xi}, \tag{61}$$

where we used the fact that $n\cdot p = n\cdot D_f$. Substituting equation (60) into the Heisenberg equation (55) and integrating in s, we obtain, after some convenient rearrangement,

$$D_f = \left[I - a\frac{A(\xi(s)) - A(\xi(0))}{\xi(s) - \xi(0)}\right]^{-1}\left\{\frac{x(s) - x(0)}{2s}\right\} + \frac{1}{4}a\sigma n\frac{F(\xi(s)) - F(\xi(0))}{\xi(s) - \xi(0)}$$
$$- \frac{n}{2n\cdot p}\left[C_1\frac{A(\xi(s)) - A(\xi(0))}{\xi(s) - \xi(0)} + \frac{1}{2}C_2\frac{F(\xi(s)) - F(\xi(0))}{\xi(s) - \xi(0)}\right] + \mathcal{O}(a^2). \tag{62}$$

Since $n^\mu a_{\mu\nu} = 0$, and keeping only terms up to first order in a, constants C_1 and C_2 are given, respectively, by

$$C_1 = D_faD_f = \left(\frac{x(s) - x(0)}{2s}\right)a\left(\frac{x(s) - x(0)}{2s}\right) + \mathcal{O}(a^2) \tag{63}$$

$$C_2 = D_fa\sigma n = \left(\frac{x(s) - x(0)}{2s}\right)a\sigma n + \mathcal{O}(a^2). \tag{64}$$

From equations (54) and (61), the proper-time Hamiltonian for the case of a Dirac particle is given by

$$H = -D_f^2, \tag{65}$$

which, in the linear approximation (weak gravitational field), can be written as

$$H = -\left(\frac{x(s)-x(0)}{2s}\right)\left[I + a\frac{A(\xi(s)) - A(\xi(0))}{\xi(s)-\xi(0)}\right]\left(\frac{x(s)-x(0)}{2s}\right). \tag{66}$$

One may repeat the steps followed in the previous section to get

$$\langle x's|x''0\rangle = \frac{\Phi(x',x'')}{s^2}\exp\left\{-\frac{\mathrm{i}}{4s}(x'-x'')\left[I + a\frac{A(\xi')-A(\xi'')}{\xi'-\xi''}\right](x'-x'')\right\}, \tag{67}$$

where $\Phi(x', x'')$ is to be determined. The constraints to be satisfied are the same as in the previous case, namely $\langle x's|p(s)|x''0\rangle = \mathrm{i}\partial'\langle x's|x''0\rangle$ and $\langle x's|p(0)|x''0\rangle = -\mathrm{i}\partial''\langle x's|x''0\rangle$. The first constraint leads to

$$\partial'\Phi = \left[\partial'\frac{\langle C_2\rangle}{4n\cdot p}(F(\xi') - F(\xi''))\right]\Phi, \tag{68}$$

where

$$\langle C_2\rangle := \langle x's|C_2|x''0\rangle = \frac{(x'-x'')}{2s}a\sigma n + \mathcal{O}(a^2). \tag{69}$$

Hence, we can write

$$\Phi(x',x'') = \chi(x'')\exp\left\{\frac{s\langle C_2\rangle(F(\xi')-F(\xi''))}{2(\xi'-\xi'')}\right\}, \tag{70}$$

where the x''-dependence on $\chi(x'')$ is determined by the second constraint written previously. The imposition of this second constraint leads to the differential equation $\partial''\chi = 0$, so that χ is a constant, denoted by C_0:

$$\Phi(x',x'') = C_0\exp\left\{\frac{s\langle C_2\rangle(F(\xi')-F(\xi''))}{2(\xi'-\xi'')}\right\}. \tag{71}$$

As before, we evaluate the remaining constant C_0 by taking the limit $s \to 0$ and the final result for $\Delta_F(x', x'')$ is written as

$$\begin{aligned}\Delta_F(x',x'') = {}& \frac{1}{16\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}m^2 s}\exp\left\{-\frac{\mathrm{i}}{4s}(x'-x'')\left[I + a\frac{A(\xi')-A(\xi'')}{\xi'-\xi''}\right](x'-x'')\right\}\\ &\times\exp\left\{\frac{(x'-x'')a\sigma n(F(\xi')-F(\xi''))}{4(\xi'-\xi'')}\right\}.\end{aligned} \tag{72}$$

The desired fermionic Green function is then obtained by inserting the last expression into equation (51).

4. Green functions by operator techniques

In this section we show that using operator techniques one can obtain the Green functions in a simple manner. We were motivated by successful calculations previously made for Green functions of relativistic charged particles in similar external electromagnetic fields [7, 8]. We shall start by discussing the case of a scalar particle. The essential idea is to start with expression (16) for the Green function, namely $G(x', x'') = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}m^2 s}\langle x's|x''0\rangle$ and then relate by an algebraic method the expression of Schrödinger-like propagator $\langle x's|x''0\rangle$

for the problem containing the interaction with the corresponding one for the free case, denoted by $\langle x's|x''0\rangle_0$. For the scalar particle we note that

$$p^2 - papF(\xi) = S_0 p^2 S_0{}^{-1}, \tag{73}$$

where we defined the operator S_0 by

$$S_0 = \exp\left[-\mathrm{i}\frac{papA(\xi)}{2n\cdot p}\right] \tag{74}$$

and used the well-known relation for operators A and B

$$\mathrm{e}^A B\,\mathrm{e}^{-A} = \sum_n C_n \qquad \text{where} \quad C_0 = B \qquad \text{and} \qquad C_{n+1} = [A, C_n]. \tag{75}$$

As a consequence, we have

$$\begin{aligned}\langle x'|\,\mathrm{e}^{\mathrm{i}s(p^2-papF)}|x''\rangle &= \langle x'|S_0\,\mathrm{e}^{\mathrm{i}sp^2}S_0{}^{-1}|x''\rangle\\ &= \langle x'|\,\mathrm{e}^{\mathrm{i}sp^2}\big(\mathrm{e}^{-\mathrm{i}sp^2}S_0\,\mathrm{e}^{\mathrm{i}sp^2}\big)S_0{}^{-1}|x''\rangle\\ &= \langle x's|S_0(s)S_0(0)^{-1}|x''0\rangle_0,\end{aligned} \tag{76}$$

where we conveniently inserted the identity operator $\mathrm{e}^{\mathrm{i}sp^2}\,\mathrm{e}^{-\mathrm{i}sp^2}$ and the meaning of the subscript 0 is such that

$$\langle x's|x''0\rangle_0 = \frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{-\mathrm{i}\frac{(x'-x'')^2}{4s}}. \tag{77}$$

Inserting (74) into (76), a straightforward calculation leads to the matrix element

$$\langle x's|x''0\rangle = \frac{1}{(4\pi s)^2}\exp\left\{-\frac{\mathrm{i}}{4s}(x'-x'')\left[I + a\frac{A(\xi')-A(\xi'')}{\xi'-\xi''}\right](x'-x'')\right\} \tag{78}$$

as an exact result. It is also the same as that found by the proper-time method in the small a approximation. This is curious. Perhaps the reason is that we have imposed very few restrictions on a. In the case of the electromagnetic field the tensor $f_{\mu\nu}$ and its dual have special properties which allow simplifications. The other difference is that the proper-time equations of motion are nonlinear.

For the Dirac particle we may proceed in an analogous way. For this purpose, we now define operator S as

$$S = \exp\left[\frac{\mathrm{i}}{4}\frac{pa\sigma n}{n\cdot p}F(\xi)\right]. \tag{79}$$

Using the BCH-like relation (75) it is straightforward to show that

$$p^2 - papF(\xi) - \frac{\mathrm{i}}{2}pa\sigma n\frac{\mathrm{d}F}{\mathrm{d}\xi} = S(p^2 - papF(\xi))S^{-1}. \tag{80}$$

Note that operator S eliminates the interaction term containing explicitly the Dirac matrices. To reduce further the remaining term can be done with the aid of operator S_0, introduced before for the spinless particle. Hence, we obtain for the Dirac case, the analogous expression of equation (76), namely

$$\langle x'|\exp\left\{\mathrm{i}s\left[p^2 - papF(\xi) - \frac{\mathrm{i}}{2}pa\sigma n\frac{\mathrm{d}F}{\mathrm{d}\xi}\right]\right\}|x''\rangle = \langle x's|S(s)S_0(s)(S(0)S_0(0))^{-1}|x''0\rangle_0. \tag{81}$$

Inserting in the previous equation the expressions of $S(s)$, $S_0(s)$, $S_0(0)^{-1}$ and $S(0)^{-1}$ it is not difficult to obtain the desired Green function for the Dirac particle.

We close this section by showing an alternative way of employing algebraic methods in order to obtain the above relativistic Green functions. As we shall see, the present method leads directly to the Green functions without using the proper-time method.

Starting with the spin-0 case, we have

$$G(x', x'') = \langle x'|G|x''\rangle, \tag{82}$$

where

$$G = (p^2 - papF(\xi) - m^2)^{-1} = S_0(p^2 - m^2)^{-1}S_0{}^{-1}, \tag{83}$$

with operator S_0 defined by equation (74). Hence, we can write

$$G_F(x', x'') = S_0(\mathrm{i}\partial')\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x'-x'')}}{p^2 - m^2 + \mathrm{i}\epsilon}S_0{}^{-1}(-\mathrm{i}\partial'') \tag{84}$$

which leads to the final result

$$G_F(x', x'') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x'-x'')}}{p^2 - m^2 + \mathrm{i}\epsilon}\exp\left\{-\mathrm{i}\frac{pap}{2n\cdot p}[A(\xi') - A(\xi'')]\right\} \tag{85}$$

For the spin $\frac{1}{2}$ the Green function $S_F(x', x'')$ is given by

$$S_F(x', x'') = \left(\mathrm{i}\gamma^\mu\partial'_\mu - \frac{\mathrm{i}}{2}F(\xi')a^{\mu\nu}\gamma_\mu\partial'_\nu + m\right)\Delta_F(x', x''), \tag{86}$$

where $\Delta_F(x', x'')$ is given by

$$\Delta_F(x', x'') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x'-x'')}}{p^2 - m^2 + \mathrm{i}\epsilon}\exp\left\{-\mathrm{i}\frac{pap}{2n\cdot p}[A(\xi') - A(\xi'')]\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{4}\frac{pa\sigma n}{n\cdot p}[F(x') - F(x'')]\right\}, \tag{87}$$

which agrees with the result obtained in section 3.

5. Ansatz solution for the Green function

In this section, we present an alternative procedure to construct previous Green functions which is based on a convenient ansatz for the desired solution. We illustrate the method for the spin-0 particle but a generalization to the spin-$\frac{1}{2}$ case can be made without difficulty. The main motivation for such a procedure is that it worked very well for the case of a relativistic charged particle under an external electromagnetic field of a plane wave [15]. The desired Green function satisfies the differential equation

$$[\partial^\mu\partial_\mu - a_{\mu\nu}F(\xi)\partial^\mu\partial^\nu + m^2]G(x - x') = -\delta(x - x'), \tag{88}$$

where, for convenience, we are using variables x and x' in this section, instead of variables x' and x'' chosen in the preceding section. Hence, along this section, x is not an operator. As usual, the Green function can be written as

$$G(x, x') = \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}m^2s}\Delta(x, x', s), \tag{89}$$

where $\Delta(x, x', s)$ satisfies the Schrödinger-like differential equation

$$[\mathrm{i}\partial_s - \partial^\mu\partial_\mu + a_{\mu\nu}F(\xi)\partial^\mu\partial^\nu]\Delta(x, x', s) = 0, \tag{90}$$

subjected to the initial condition

$$\mathrm{i}\Delta(x, x', s) \xrightarrow{s\to 0^+} \delta(x - x'). \tag{91}$$

Now, in order to factorize the free-particle solution, we try the following ansatz:

$$\Delta(x, x', s) = \Delta_0(x, x', s)\Sigma(x, x', s), \tag{92}$$

where $\Delta_0(x, x')$ corresponds to the free-particle solution, that is,

$$(\mathrm{i}\partial_s - \partial^2)\Delta_0(x, x') = 0, \tag{93}$$

whose the well-known solution is given by

$$\Delta_0(x, x', s) = \frac{\mathrm{i}}{16\pi^2 s^2} \exp\left\{-\mathrm{i}\frac{(x-x')^2}{4s}\right\}. \tag{94}$$

Substituting our ansatz (92) into equation (90), we have

$$\Delta_0 \mathrm{i}\partial_s \Sigma - 2\partial_\mu \Delta_0 \partial^\mu \Sigma - \Delta_0 \partial^2 \Sigma + F a_{\mu\nu} \left(\partial^\mu \partial^\nu \Delta_0\right) \Sigma \\ + 2F a_{\mu\nu} \partial^\mu \Delta_0 \partial^\nu \Sigma + F a_{\mu\nu} \left(\partial^\mu \partial^\nu \Sigma\right) \Delta_0 = 0. \tag{95}$$

Since $h_{\mu\nu}$ is small, we neglect the last two terms in the above equation and get

$$\Delta_0 \mathrm{i}\partial_s \Sigma - 2\partial_\mu \Delta_0 \partial^\mu \Sigma - \Delta_0 \partial^2 \Sigma + F a_{\mu\nu} \left(\partial^\mu \partial^\nu \Delta_0\right) \Sigma = 0. \tag{96}$$

Next, using that

$$\begin{aligned} \partial_\mu \Delta_0 &= -\mathrm{i}\frac{(x-x')_\mu}{2s}\Delta_0 \\ \partial_\mu \partial_\nu \Delta_0 &= -\mathrm{i}\frac{\eta_{\mu\nu}}{2s}\Delta_0 - \frac{(x-x')_\mu (x-x')_\nu}{4s^2}\Delta_0, \end{aligned} \tag{97}$$

equation (96) takes the form

$$\left[\mathrm{i}\partial_s - \partial_\mu \partial^\mu + \mathrm{i}\frac{(x-x')_\mu}{s}\partial^\mu - \frac{F}{4s^2}(x-x')_\mu a^{\mu\nu}(x-x')_\nu\right]\Sigma = 0. \tag{98}$$

An inspection of the previous equation suggests us to try a solution of the form

$$\Sigma = \exp\left\{\frac{-\mathrm{i} f(\xi, \xi')(x-x')_\mu a^{\mu\nu}(x-x')_\nu}{4s}\right\}. \tag{99}$$

In order to substitute the last expression into equation (98), we need

$$\mathrm{i}\partial_s \Sigma = -f\frac{(x-x')_\mu a^{\mu\nu}(x-x')_\nu}{4s^2}\Sigma \tag{100}$$

and

$$\partial_\mu \Sigma = -\frac{\mathrm{i}}{4s}\left[n_\mu \frac{\mathrm{d}f}{\mathrm{d}\xi}(x-x')_\mu a^{\mu\nu}(x-x')_\nu + 2f a_{\mu\nu}(x-x')^\nu\right]\Sigma. \tag{101}$$

Further, we also have

$$\partial_\mu \partial^\mu \Sigma = -\frac{1}{16s^2}\left[n_\mu \frac{\mathrm{d}f}{\mathrm{d}\xi}(x-x')_\alpha a^{\alpha\beta}(x-x')_\beta + 2f a_{\mu\alpha}(x-x')^\alpha\right] \\ \times \left[n^\mu \frac{\mathrm{d}f}{\mathrm{d}\xi}(x-x')_\alpha a^{\alpha\beta}(x-x')_\beta + 2f a^{\mu\alpha}(x-x')_\alpha\right]\Sigma, \tag{102}$$

where we have used the properties of n_μ and $a_{\mu\nu}$, namely $n^2 = 0$ and $n_\mu a^{\mu\nu} = 0$. To the order of magnitude of our interest we may write

$$\partial_\mu \partial^\mu \Sigma = 0. \tag{103}$$

Hence, substituting equations (103), (101) and (100) into equation (98), we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[(\xi - \xi')f] = F. \tag{104}$$

We have then

$$f(\xi,\xi') = \frac{A(\xi) - A(\xi')}{\xi - \xi'}. \tag{105}$$

where we defined function A by

$$F = \frac{\mathrm{d}A}{\mathrm{d}\xi}. \tag{106}$$

Therefore, equation (92) takes the form

$$\begin{aligned}\Delta(x,x',s) &= \frac{\mathrm{i}}{16\pi^2 s^2}\exp\left\{-\mathrm{i}\frac{(x-x')^2 + (x-x')^\mu a_{\mu\nu}\Omega(\xi)(x-x')^\nu}{4s}\right\}\\ &= \frac{\mathrm{i}}{16\pi^2 s^2}\exp\left\{-\mathrm{i}\frac{(x-x')^2 + (x-x')^\mu a_{\mu\nu}\frac{A(\xi)-A(\xi')}{\xi-\xi'}(x-x')^\nu}{4s}\right\}.\end{aligned} \tag{107}$$

With the purpose of checking the self-consistency of the result just obtained, not that

$$\lim_{s\to 0^+}\int \mathrm{d}^4k\,\mathrm{e}^{\mathrm{i}k^\mu(B_{\mu\nu}k^\nu s - \Delta x_\mu)} = \lim_{s\to 0^+}\exp\left\{-\mathrm{i}\frac{\Delta x_\mu (B^{-1})^{\mu\nu}\Delta x_\nu}{4s}\right\}\frac{\mathrm{i}(\pi)^2}{s^2\sqrt{\det(B)}}. \tag{108}$$

As a consequence, we have

$$\lim_{s\to 0^+}\frac{\mathrm{i}}{16\pi^2 s^2}\exp\left\{-\mathrm{i}\frac{\Delta x_\mu (B^{-1})^{\mu\nu}\Delta x_\nu}{4s}\right\} = \delta^4(x-x')\sqrt{\det(B)}. \tag{109}$$

Making the following identifications:

$$\left(B^\mu_\nu\right)^{-1} = \delta^\mu_\nu + a^\mu_\nu\frac{\Delta A}{\Delta\xi} \qquad B^\mu_\nu = \delta^\mu_\nu - a^\mu_\nu\frac{\Delta A}{\Delta\xi}$$

we get

$$\det(B) = 1 + \mathrm{Tr}\left(a^\mu_\nu\right)\frac{A(\xi) - A(\xi')}{\xi - \xi'} = 1, \tag{110}$$

which confirms the initial condition in the parameter s,

$$\mathrm{i}\Delta(x,x',s) \overset{s\to 0^+}{\longrightarrow} \delta(x-x'). \tag{111}$$

Inserting expression (107) into equation (89), we obtain the desired Green function, in agreement with our previous calculations.

6. Semiclassical approximation

As we have seen in many of the previous calculations, relativistic Green functions may be written in terms of Schrödinger-like propagators, if we introduce appropriately an integration over the so-called proper-time s. For instance, the Green function for a scalar particle can be written as $G_F(x',x'') = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}s(m^2-\mathrm{i}\epsilon)}\langle x's|x''0\rangle$ (see equation (7)), where $\langle x's|x''0\rangle$ can be interpreted as a Feynman propagator of an auxiliary problem of a non-relativistic particle in four dimensions whose dynamics corresponds to the evolution in the parameter s, which plays the role of time in this auxiliary problem. Once $\langle x's|x''0\rangle$ behaves like a non-relativistic Feynman propagator, we have at our disposal all techniques developed to compute this quantity as, for example, the Feynman path integral method. In particular, in the context of path integrals, it may be convenient to use the semiclassical approximation. It is well known that whenever the corresponding Lagrangian is quadratic in the coordinates and velocities the semiclassical result gives the exact result.

The purpose of this section is to show that if we apply the semiclassical approximation to the problem at hand (a relativistic particle in a plane-wave gravitational background) we shall obtain the exact result. At first sight this is an unexpected result, since the Lagrangian of the corresponding classical problem is far from being quadratic. However, there is a strong reason that suggests that this will be the case, namely the semiclassical method when applied to the problem of a relativistic charged particle in an external field of an electromagnetic plane wave yields the exact result (see, for instance, [26]). Here, we shall discuss only the case of a scalar particle. To avoid any confusion with the notation, observe that, in this section, all quantities are not operators, but classical numbers. The semiclassical approximation for the Feynman propagator $\langle x's|x''0\rangle$ is given by

$$\langle x's|x''0\rangle = \frac{1}{(2\pi \mathrm{i})^2}\left|\frac{\partial^2 S_{\mathrm{cl}}}{\partial x'\partial x''}\right|^{1/2} \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}}, \tag{112}$$

where S_{cl} means the functional action

$$S(x_\mu) = \int_0^s L(x_\mu(\tau), \dot{x}_\mu(\tau))\,\mathrm{d}\tau \tag{113}$$

evaluated with the classical solution x^μ_{cl}, that is, $S_{\mathrm{cl}} = S(x_{\mathrm{cl}})$, where x_{cl} satisfies the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}_\mu}\right)\bigg|_{x^\mu = x^\mu_{\mathrm{cl}}} = \frac{\partial L}{\partial x_\mu}\bigg|_{x^\mu = x^\mu_{\mathrm{cl}}}, \qquad \mu = 0, 1, 2, 3 \tag{114}$$

submitted to the Feynman conditions

$$x^\mu_{\mathrm{cl}}(\tau = 0) = x''^\mu, \qquad x^\mu_{\mathrm{cl}}(\tau = s) = x'^\mu. \tag{115}$$

Hence, in order to use this approximation, we need to construct the classical Lagrangian corresponding to the following classical Hamiltonian:

$$H(x, p) = -p^2 + p^\mu h_{\mu\nu}(x) p^\nu. \tag{116}$$

Recalling that $p_\mu = \partial L/\partial \dot{x}^\mu$, it is not difficult to show that the corresponding Lagrangian can be written as

$$L(x, \dot{x}) = -\tfrac{1}{4}(\dot{x}_\mu \dot{x}^\mu + \dot{x}^\mu h_{\mu\nu}(x)\dot{x}^\nu) = -\tfrac{1}{4} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{117}$$

The functional action is then given by

$$S(x) = \int_0^s \left[-\frac{1}{4}\dot{x}^\mu(\tau)\dot{x}_\mu(\tau) - \frac{1}{4}\dot{x}^\mu(\tau) a_{\mu\nu}\dot{x}^\nu(\tau) F(n\cdot x(\tau))\right]\mathrm{d}\tau. \tag{118}$$

The application of Euler–Lagrange equations (114) to the Lagrangian (117) leads to classical equations completely analogous to the Heisenberg equations discussed in section 2, but does not forget that here x_μ and p_μ are not operators. However, exactly the same kind of solutions is obtained from these equations, so that we just write the classical solution as

$$x^\mu_{\mathrm{cl}}(\tau) = x^\mu_{\mathrm{cl}}(0) + \left[\frac{n^\mu C}{2(n\cdot p)^2} - \frac{a^{\mu\nu} D_\nu}{n\cdot p}\right][A(\xi(\tau)) - A(\xi(0))] - 2D^\mu \tau, \tag{119}$$

where C and D_μ are (classical) constants to be determined by imposing Feynman conditions (115). From the classical version of (38) we have, up to first order in a,

$$C = \frac{1}{4}\frac{(x' - x'')_\mu a^{\mu\nu}(x' - x'')_\nu}{s^2} =: \frac{1}{4}\frac{\Delta x^\mu a_{\mu\nu}\Delta x^\nu}{s^2}, \tag{120}$$

where we defined $\Delta x^\mu = (x' - x'')^\mu$.

In order to obtain D^μ, we take $\tau = s$ in equation (119) and contract $x^\mu_{\mathrm{cl}}(s)$ with n_μ and $a^{\mu\nu}$ to obtain, respectively,

$$n^\mu \Delta x_\mu = -2D^\mu n_\mu s \quad \Longrightarrow \quad n\cdot D = -\frac{\Delta\xi}{2s}; \tag{121}$$

$$a^{\mu\nu}\Delta x_\nu = -2D_\nu a^{\mu\nu}s, \quad \Longrightarrow \quad a^{\mu\nu}D_\nu = -\frac{a^{\mu\nu}\Delta x_\nu}{2s}, \tag{122}$$

where we defined $\Delta\xi = n^\mu(x'-x'')_\mu = \xi' - \xi''$. Substituting these relations into solution (119) with $\tau = s$, we obtain D^μ in terms of x' and x'', namely

$$D^\mu = \frac{1}{2s}\left(n^\mu \frac{\Delta x_\alpha a^{\alpha\beta}\Delta x_\beta}{2(\Delta\xi)^2} + \frac{a^{\mu\nu}\Delta x_\nu}{\Delta\xi}\right)\Delta A - \frac{\Delta x^\mu}{2s}, \tag{123}$$

where we $\Delta A := A(\xi') - A(\xi'')$. Substituting expressions (120) and (123) into equation (119), and then differentiating with respect to τ, we obtain the classical velocity $\dot{x}^\mu_{\rm cl}(\tau)$ at any 'instant' τ in terms of x' and x'', namely

$$\dot{x}^\mu(\tau) = \left(n^\mu \frac{\Delta x_\alpha a^{\alpha\beta}\Delta x_\beta}{2(\Delta\xi)^2} + \frac{a^{\mu\nu}\Delta x_\nu}{\Delta\xi}\right)\frac{\mathrm{d}A}{\mathrm{d}\tau} - 2D^\mu. \tag{124}$$

Substituting the last equation into the functional action (118) and keeping only terms up to first order in a, we obtain after a lengthy but straightforward calculation the desired classical action,

$$\begin{aligned} S_{\rm cl} &= -\frac{1}{4s}\Delta x_\mu\left(\eta^{\mu\nu} + a^{\mu\nu}\frac{\Delta A}{\Delta\xi}\right)\Delta x_\nu \\ &= -\frac{1}{4s}(x'-x'')^\mu\left(\eta_{\mu\nu} + a_{\mu\nu}\frac{A(\xi') - A(\xi'')}{\xi' - \xi''}\right)(x'-x'')^\nu. \end{aligned}$$

Now we need to evaluate the Van Vleck–Pauli–Morette determinant to obtain the pre-exponential factor of the semiclassical propagator. With this purpose, first note that

$$\frac{\partial^2 S_{\rm cl}}{\partial x'_\mu \partial x''_\nu} = \frac{1}{2s}(\eta^{\mu\nu} + \mathcal{O}(a)). \tag{125}$$

As a consequence, we have

$$\det\left(\frac{\partial^2 S_{\rm cl}}{\partial x' \partial x''}\right) = \frac{1}{16s^4}\det(\eta + \mathcal{O}(a)) = \frac{1}{16s^4}(1 + \mathrm{Tr}\mathcal{O}(a)). \tag{126}$$

However, note that $\mathrm{Tr}(\mathcal{O}(a)) = \mathcal{O}(a^2)$ so that $\det^{1/2}\left(\frac{\partial^2 S_{\rm cl}}{\partial x'\partial x''}\right) = \frac{1}{4s^2} + \mathcal{O}(a^2)$. Collecting the previous results and using equation (112) we conclude that, up to first order in a, the Feynman propagator $\langle x's|x''0\rangle$ in the semiclassical approximation is given by

$$\begin{aligned} \langle x's|x''0\rangle &= \frac{1}{16\pi^2 s^2}\exp\left\{\frac{-\mathrm{i}}{4s}\Delta x_\mu\left(\eta^{\mu\nu} + a^{\mu\nu}\frac{\Delta A}{\Delta\xi}\right)\Delta x_\nu\right\} \\ &= \frac{1}{16\pi^2 s^2}\exp\left\{\frac{-\mathrm{i}}{4s}(x'-x'')^\mu\left(\eta_{\mu\nu} + a_{\mu\nu}\frac{A(\xi') - A(\xi'')}{\xi' - \xi''}\right)(x'-x'')^\nu\right\}. \end{aligned} \tag{127}$$

Substituting the last expression into equation (16), we reobtain the correct Green function for a scalar relativistic particle in the weak gravitational field of a plane wave.

7. Conclusions and final remarks

In this work we have presented a few alternative techniques to calculate Green functions of relativistic particles under the influence of a weak gravitational field of a plane wave, with particular attention to the Fock–Schwinger proper-time method. In fact, we started this paper by a detailed application of this method in the construction of the Green functions for both spin-0 as well as spin-$\frac{1}{2}$ particles. We showed how these Green functions can be obtained by algebraic methods in an extremely compact and elegant way. We also showed how to reobtain these solutions with an appropriate ansatz for the Green function and, finally, we discussed a semiclassical solution and checked that for the case at hand this approximation is sufficient to give the exact result. From one hand, this is a surprising result, since the Lagrangian involved in the solution is far from being quadratic. On the other hand, it could have been guessed, since a semiclassical approximation yields the exact Green function for a relativistic charged particle in an external electromagnetic field of a plane wave. The results obtained here corroborate and complement those of [13].

It should be emphasized that the main purpose of the present work is to provide the reader with alternative methods of computing Green functions or, better saying, to popularize some powerful methods of calculation (like Schwinger's method) that have rarely been used in the literature. Our emphasis was in great part on the methods themselves and not on the problems. Though we have not applied the above-mentioned methods to new situations, we think our calculations may be useful in the study of unsolved problems. Particularly, the search for other non-quadratic problems whose exact solutions coincide with those obtained by the semiclassical approximation is an interesting issue. For the problem at hand, namely relativistic particles in a plane-wave gravitational background, it is difficult to give a satisfactory explanation why it occurs. What we can say is that, as it happens for the electromagnetic case, it is due to the peculiarities of a plane-wave field. We think it is extremely difficult to anticipate whether a semiclassical approach when applied to a non-quadratic problem will lead to the exact answer. In fact, the number of problems for which the semiclassical approach yields the exact result is so few that any new problem with this property is extremely welcome. Maybe after this list of peculiar problems is enlarged enough we will be able to establish a concrete correlation among them and will finally understand the ultimate meaning of the semiclassical approach.

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